



## Kurepa-trees and Namba-forcing

Bernhard Koenig, Yasuo Yoshinobu

### ► To cite this version:

| Bernhard Koenig, Yasuo Yoshinobu. Kurepa-trees and Namba-forcing. 2006. hal-00023751

**HAL Id: hal-00023751**

**<https://hal.science/hal-00023751>**

Preprint submitted on 4 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# KUREPA-TREES AND NAMBA FORCING

BERNHARD KÖNIG AND YASUO YOSHINOBU

**ABSTRACT.** We show that compact cardinals and MM are sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ . This is done by adding ‘regressive’  $\lambda$ -Kurepa-trees in either case. We argue that the destruction of regressive Kurepa-trees with MM requires the use of Namba forcing.

## 1. INTRODUCTION

Say that a tree  $T$  of height  $\lambda$  is  $\gamma$ -*regressive* if for all limit ordinals  $\alpha < \lambda$  with  $\text{cf}(\alpha) < \gamma$  there is a function  $f_\alpha : T_\alpha \rightarrow T_{<\alpha}$  which is *regressive*, i.e.  $f_\alpha(x) <_T x$  for all  $x \in T_\alpha$  and if  $x, y \in T_\alpha$  are distinct then  $f_\alpha(x)$  or  $f_\alpha(y)$  is strictly above the meet of  $x$  and  $y$ . We give a summary of the main results of this paper:

**5 Theorem.** *For all uncountable regular  $\lambda$  there is a  $\lambda$ -closed forcing  $\mathcal{K}_{\text{reg}}^\lambda$  that adds a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree.*

This is contrasted in Section 4:

**7 Theorem.** *Assume that  $\kappa$  is a compact cardinal and  $\lambda \geq \kappa$  is regular. Then there are no  $\kappa$ -regressive  $\lambda$ -Kurepa-trees.*

Theorems 5 and 7 establish that compact cardinals are sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ . This should be compared with the well-known result that a supercompact cardinal  $\kappa$  can be made indestructible by  $\kappa$ -directed-closed forcings [10]. These results drive a major wedge between the notions of  $\lambda$ -closed and  $\lambda$ -directed-closed. Another contrasting known result is that a strong cardinal  $\kappa$  can be made indestructible by  $\kappa^+$ -closed forcings [3]. In Section 7 we prove

**13 Theorem.** *Under MM, there are no  $\omega_1$ -regressive  $\lambda$ -Kurepa-trees for any uncountable regular  $\lambda$ .*

This shows that MM is sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ , thus answering a question from both [7] and [8]. Note that MM is indestructible by  $\omega_2$ -directed-closed forcings [8], so again we

---

2000 *Mathematics Subject Classification.* 03E40, 03E55.

*Key words and phrases.* Kurepa trees, compact cardinals, Martin’s Maximum.

find a remarkable gap between the notions of  $\omega_2$ -closed and  $\omega_2$ -directed-closed. Interestingly enough though,  $\omega_2$ -closed forcings can only violate a very small fragment of MM. To see this, let us denote by  $\Gamma_{\text{cov}}$  the class of posets that preserve stationary subsets of  $\omega_1$  and have the *covering property*, i.e. every countable set of ordinals in the extension can be covered by a countable set in the ground model. Then we have the following result from [7, p.302]:

**1 Theorem.** *The axioms PFA,  $\text{MA}(\Gamma_{\text{cov}})$  and  $\text{MA}^+(\Gamma_{\text{cov}})$  are all indestructible by  $\omega_2$ -closed forcings respectively.<sup>1</sup>*

So Theorem 5 gives

**2 Corollary.** *If  $\lambda \geq \omega_2$  is regular, then  $\text{MA}^+(\Gamma_{\text{cov}})$  is consistent with the existence of a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree.*

Again, compare this with Theorem 13. It is interesting to add that  $\text{MA}^+(\Gamma_{\text{cov}})$  in particular implies the axioms  $\text{PFA}^+$  and SRP. The typical example of a forcing that preserves stationary subsets of  $\omega_1$  but does not have the covering property is Namba forcing and the proofs confirm that Namba forcing plays a crucial role in this context. It has already been established in [9] and [11] that  $\text{MA}(\Gamma_{\text{cov}})$  can be preserved in an  $(\omega_1, \infty)$ -distributive forcing extension in which the Namba-fragment of MM fails. In our case though, the failure of MM is obtained with a considerably milder forcing, i.e.  $\lambda$ -closed for arbitrarily large  $\lambda$ .

The authors would like to thank Yoshihiro Abe, Tadatashi Miyamoto and Justin Moore for their helpful comments.

The reader requires a strong background in set-theoretic forcing, a good prerequisite would be [4]. We give some definitions that might not be in this last reference or because we defined them in a slightly different fashion. If  $\Gamma$  is a class of posets then  $\text{MA}(\Gamma)$  denotes the statement that whenever  $\mathcal{P} \in \Gamma$  and  $D_\xi$  ( $\xi < \omega_1$ ) is a collection of dense subsets of  $\mathcal{P}$  then there exists a filter  $G$  on  $\mathcal{P}$  such that  $D_\xi \cap G \neq \emptyset$  for all  $\xi < \omega_1$ . The stronger  $\text{MA}^+(\Gamma)$  denotes the statement that whenever  $\mathcal{P} \in \Gamma$ ,  $D_\xi$  ( $\xi < \omega_1$ ) are dense subsets of  $\mathcal{P}$ , and  $\dot{S}$  is a  $\mathcal{P}$ -name such that

$$\Vdash_{\mathcal{P}} \dot{S} \text{ is stationary in } \omega_1$$

then there exists a filter  $G$  on  $\mathcal{P}$  such that  $D_\xi \cap G \neq \emptyset$  for all  $\xi < \omega_1$ , and

$$\dot{S}[G] = \{\gamma < \omega_1 : \exists q \in G (q \Vdash_{\mathcal{P}} \check{\gamma} \in \dot{S})\}$$

is stationary in  $\omega_1$ . In particular, PFA is  $\text{MA}(\text{proper})$  and MM is  $\text{MA}(\text{preserving stationary subsets of } \omega_1)$ . The interested reader is referred to [1] and [2] for the history of these *forcing axioms*.

---

<sup>1</sup>See below for a definition of the axioms  $\text{MA}(\Gamma)$  and  $\text{MA}^+(\Gamma)$ .

A partial order is  $\lambda$ -closed if it is closed under descending chains of length less than  $\lambda$ . It is  $\lambda$ -directed-closed if it is closed under directed subsets of size less than  $\lambda$ . [7] proves that PFA is preserved by  $\omega_2$ -closed forcings and [8] that MM is preserved by  $\omega_2$ -directed-closed forcings.

*Namba forcing* is denoted by  $\text{Nm}$ : conditions are trees  $t \subseteq \omega_2^{<\omega}$  with a *trunk*  $\text{tr}(t)$  such that  $t$  is linear below  $\text{tr}(t)$  and has splitting  $\aleph_2$  everywhere above the trunk. Smaller trees contain more information. It is known that Namba forcing preserves stationary subsets of  $\omega_1$ . If  $t \in \text{Nm}$  and  $x \in t$  then the last element of  $x$  is also called the *tag* of  $x$ , denoted as  $\text{tag}(x)$ , and we define  $\text{Suc}_t(x)$  to be the set of tags of all immediate successors of  $x$  in  $t$ . So  $\text{Suc}_t(x)$  is an unbounded subset of  $\omega_2$ . In an abuse of notation, a sequence is sometimes confused with its tag. We write  $[t]$  for the set of infinite branches through  $t$ .

## 2. STATIONARY LIMITS

For a tree  $T$  and an ordinal  $\alpha$ , let  $T_\alpha$  denote the  $\alpha$ th level of  $T$  and  $T_{<\alpha} = \bigcup_{\xi < \alpha} T_\xi$ . If  $X$  is a set of ordinals, we write  $T \restriction X$  for the subtree  $\bigcup_{\xi \in X} T_\xi$ . The expression  $\text{ht}(T)$  denotes the *height* of  $T$ . We only consider trees of functions. If  $T$  is a tree and  $\mathcal{B}$  a collection of cofinal branches through  $T$  then we call  $\mathcal{B}$  *non-stationary over  $T$*  if there is a function  $f : \mathcal{B} \rightarrow T$  which is *regressive*, i.e.  $f(b) \in b$  for all  $b \in \mathcal{B}$  and if  $b, b' \in \mathcal{B}$  are distinct then  $f(b)$  or  $f(b')$  is strictly above  $b \cap b'$ . Otherwise we call  $\mathcal{B}$  *stationary over  $T$* . A tree  $T$  of height  $\kappa$  is called  $\gamma$ -*regressive* if  $T_\alpha$  is non-stationary over  $T_{<\alpha}$  for every limit ordinal  $\alpha < \kappa$  of cofinality less than  $\gamma$ . The following is easy to check:

**3 Remark.** Assume that  $A \subseteq \alpha$  is cofinal in  $\alpha$ . Then  $T_\alpha$  is stationary over  $T_{<\alpha}$  iff  $T_\alpha$  is stationary over  $T \restriction A$ .

The  $\omega$ -cofinal limits will figure prominently when dealing with  $\omega_1$ -regressive trees, so we prove a useful Lemma about these. For simplicity we only consider trees of height  $\omega$ . The reader will notice that the following observations are applicable in Section 6. If  $T$  is of height  $\omega$  and  $\mathcal{B}$  a collection of infinite branches then for any subset  $S \subseteq T$  we let

$$\overline{S} = \{b \in \mathcal{B} : b \cap S \text{ is infinite}\}.$$

If  $S$  is countable and  $\overline{S}$  uncountable then we call  $S$  a *Cantor-subtree* of  $T$ . The class  $\mathcal{N}(T, \mathcal{B}) \subseteq [H_\theta]^{\aleph_0}$  (for some large enough regular  $\theta$ ) is defined by letting  $N \in \mathcal{N}(T, \mathcal{B})$  if and only if there is  $b \in \mathcal{B}$  such that  $b \subseteq N$  but  $b \notin N$ . We have the following

**4 Lemma.** Assume that  $T$  has height  $\omega$  and size  $\aleph_1$  and that  $\mathcal{B}$  is a collection of infinite branches. Then the following are equivalent:

- (1)  $\mathcal{B}$  is stationary over  $T$ .
- (2) (a) Either there is a Cantor-subtree  $S \subseteq T$  or  
 (b) if we identify  $T$  with  $\omega_1$  by any enumeration then

$$E_{\mathcal{B}} = \{\alpha < \omega_1 : \sup(b) = \alpha \text{ for some } b \in \mathcal{B}\}$$

is stationary in  $\omega_1$ .

- (3)  $\mathcal{N}(T, \mathcal{B})$  is stationary in  $[H_\theta]^{\aleph_0}$ .

*Proof.* The equivalence of (1) and (3) can be found in [6, p.112] and the implication (2)  $\implies$  (1) is easy.

For (3)  $\implies$  (2), assume  $\neg(2)$  and show  $\neg(3)$ : pick an enumeration  $e : \omega_1 \rightarrow T$  such that  $E_{\mathcal{B}}$  is nonstationary if we identify nodes with countable ordinals via the enumeration  $e$ . Pick a structure  $N \prec H_\theta$  such that  $e, T, \mathcal{B} \in N$  and set  $\gamma = N \cap \omega_1$ , so we have  $\gamma \notin E_{\mathcal{B}}$ . Let  $b \in \mathcal{B}$  be such that  $b \subseteq N$ . Then  $\sup(b) < \gamma$  holds. Now define

$$\mathcal{A} = \{c \in \mathcal{B} : \sup(c) = \sup(b)\}.$$

Note that  $\mathcal{A} \in N$  and  $\mathcal{A}$  is countable since we know by  $\neg(2)(a)$  that  $\overline{\sup(b)}$  is countable. So  $\mathcal{A} \subseteq N$ , therefore  $b \in N$ . This shows that  $N \notin \mathcal{N}(T, \mathcal{B})$  and  $\mathcal{N}(T, \mathcal{B})$  is non-stationary.  $\square$

Note that the equivalence of (1) and (3) is to some extent already in [1, p.955] but our result differs slightly from this last reference as we have a stronger notion of non-stationarity. See also [6] for variations of Lemma 4 in uncountable heights.

### 3. CREATING REGRESSIVE KUREPA-TREES

Let  $\lambda$  be a regular uncountable cardinal throughout this section. We describe the natural forcing  $\mathcal{K}_{\text{reg}}^\lambda$  to add a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree and show that this forcing is  $\lambda$ -closed. We may assume the cardinal arithmetic  $2^{<\lambda} = \lambda$ , otherwise a preliminary Cohen-subset of  $\lambda$  could be added. Conditions of  $\mathcal{K}_{\text{reg}}^\lambda$  are pairs  $(T, h)$ , where

- (1)  $T$  is a tree of height  $\alpha + 1$  for some  $\alpha < \lambda$  and each level has size  $< \lambda$ .
- (2)  $T$  is  $\lambda$ -regressive, i.e. if  $\xi \leq \alpha$  then  $T_\xi$  is non-stationary over  $T_{<\xi}$ .
- (3)  $h : T_\alpha \longrightarrow \lambda^+$  is 1-1.

The condition  $(T, h)$  is stronger than  $(S, g)$  if

- $S = T \upharpoonright \text{ht}(S)$ .
- $\text{rng}(g) \subseteq \text{rng}(h)$ .
- $g^{-1}(\nu) \leq_T h^{-1}(\nu)$  for all  $\nu \in \text{rng}(g)$ .

A generic filter  $G$  for  $\mathcal{K}_{\text{reg}}^\lambda$  will produce a  $\lambda$ -regressive  $\lambda$ -tree  $T_G$  in the first coordinate and the sets

$$b_\nu = \{x \in T_G : \text{there is } (T, h) \in G \text{ such that } h(x) = \nu\}$$

for  $\nu < \lambda^+$  form a collection of  $\lambda^+$ -many mutually different  $\lambda$ -branches through the tree  $T_G$ . Notice also that the standard arguments for  $\lambda^+$ -cc go through here as we assumed  $2^{<\lambda} = \lambda$ .

So we are done once we show that  $\mathcal{K}_{\text{reg}}^\lambda$  is  $\lambda$ -closed. To this end, let  $(T^\xi, h^\xi)$  ( $\xi < \gamma$ ) be a descending chain of conditions of length less than  $\lambda$ . We can obviously assume that  $\gamma$  is a limit ordinal. If the height of  $T^\xi$  is  $\alpha^\xi + 1$ , let  $\alpha^\gamma = \sup_{\xi < \gamma} \alpha^\xi$ . We want to extend the tree

$$T^* = \bigcup_{\xi < \gamma} T^\xi,$$

so we have to define the  $\alpha^\gamma$ th level: whenever  $\nu \in \text{rng}(h^\xi)$  for some  $\xi < \gamma$ , then there is exactly one  $\alpha^\gamma$ -branch  $c_\nu$  that has color  $\nu$  on a final segment. Now define

$$T_{\alpha^\gamma}^\gamma = \{c_\nu : \nu \in \text{rng}(h^\xi) \text{ for some } \xi < \gamma\}$$

and let  $T^\gamma$  be the tree  $T^*$  with the level  $T_{\alpha^\gamma}^\gamma$  on top. The 1-1 function  $h^\gamma : T_{\alpha^\gamma}^\gamma \longrightarrow \lambda^+$  is defined by letting

$$h^\gamma(c_\nu) = \nu.$$

We claim that  $(T^\gamma, h^\gamma)$  is a condition: the only thing left to check is that  $T_{\alpha^\gamma}^\gamma$  is non-stationary over  $T^*$ . But this is witnessed by the function

$$f(c_\nu) = \text{the } <_T\text{-least } x \in c_\nu \text{ such that there is } \xi < \gamma \text{ with } h^\xi(x) = \nu.$$

Notice that  $f$  is regressive: if

$$f(c_\nu) \leq_T f(c_\mu) \leq_T c_\nu \cap c_\mu,$$

let  $\xi$  witness that  $f(c_\mu) = x$ , i.e.  $h^\xi(x) = \mu$ . Then  $h^\xi(x)$  must be color  $\nu$  as well since  $f(c_\nu) \leq_T x$  has color  $\nu$ . Thus,  $\nu = h^\xi(x) = \mu$ .

But  $(T^\gamma, h^\gamma)$  extends the chain  $(T^\xi, h^\xi)$  ( $\xi < \gamma$ ), so we just showed

**5 Theorem.**  $\mathcal{K}_{\text{reg}}^\lambda$  is a  $\lambda$ -closed forcing that adds a  $\lambda$ -regressive  $\lambda$ -Kurepa-tree.

We emphasize again that the forcing  $\mathcal{K}_{\text{reg}}^\lambda$  is *not*  $\omega_2$ -directed-closed but the reader can check that the usual forcing to add a plain  $\lambda$ -Kurepa-tree (see e.g. [4]) actually *is*  $\lambda$ -directed-closed.

#### 4. DESTROYING REGRESSIVE KUREPA-TREES ABOVE A COMPACT CARDINAL

If  $\lambda$  is a regular uncountable cardinal then a tree  $T$  is called a *weak  $\lambda$ -Kurepa-tree* if

- $T$  has height  $\lambda$ ,
- each level has size  $\leq \lambda$  and
- $T$  has  $\lambda^+$ -many cofinal branches.

**6 Lemma.** *Suppose that  $\lambda$  is a regular uncountable cardinal and there is an elementary embedding  $j : V \longrightarrow M$  such that  $\eta = \sup(j''\lambda) < j(\lambda)$  and  $\text{cf}^M(\eta) < j(\kappa)$ . Then there are no  $\kappa$ -regressive weak  $\lambda$ -Kurepa-trees.*

*Proof.* Suppose that  $T$  is a  $\kappa$ -regressive weak  $\lambda$ -Kurepa-tree and  $j$  as above. Then there is a regressive function  $f_\eta$  defined on the level  $(jT)_\eta$ . If  $b$  is a cofinal branch through  $T$ , then we find  $\alpha_b < \lambda$  such that

$$f_\eta(jb \restriction \eta) \leq_{jT} jb \restriction j(\alpha_b) = j(b \restriction \alpha_b).$$

Note that if  $b$  and  $b'$  are two distinct branches through  $T$  then  $jb$  and  $jb'$  must disagree below  $\eta$ . Moreover,  $j(b \restriction \alpha_b) \neq j(b' \restriction \alpha_{b'})$  holds because  $f_\eta$  is regressive. Then the assignment  $b \longmapsto b \restriction \alpha_b$  must be 1-1, which is a contradiction to the fact that  $T$  has  $\lambda^+$ -many branches.  $\square$

Recall that a cardinal  $\kappa$  is  $\lambda$ -compact if there is a fine ultrafilter on  $\mathcal{P}_\kappa\lambda$ . If  $\lambda$  is regular, the elementary embedding  $j : V \longrightarrow M$  with respect to such a fine ultrafilter has the following properties:

- the critical point of  $j$  is  $\kappa$ ,
- there is a discontinuity at  $\lambda$ , i.e.  $\eta = \sup(j''\lambda) < j(\lambda)$  and
- $\text{cf}^M(\eta) < j(\kappa)$ .

(see [5, §22] for more details). A cardinal  $\kappa$  is said to be *compact* if it is  $\lambda$ -compact for all  $\lambda$ , so it follows from Lemma 6 and the above definition:

**7 Theorem.** *Assume that  $\kappa$  is a compact cardinal and  $\lambda \geq \kappa$  is regular. Then there are no  $\kappa$ -regressive weak  $\lambda$ -Kurepa-trees.*

Using Theorem 5, we have

**8 Corollary.** *Compact cardinals are sensitive to  $\lambda$ -closed forcings for arbitrarily large  $\lambda$ .*

It was known before that adding a slim<sup>2</sup>  $\kappa$ -Kurepa-tree destroys the ineffability of  $\kappa$  and that slim  $\kappa$ -Kurepa-trees can be added with  $\kappa$ -closed forcing. But note that our notion of regressive is more universal: slim Kurepa-trees can exist above compact or even supercompact cardinals.

## 5. OSCILLATING BRANCHES

Now assume that  $T$  is an  $\omega_2$ -tree: we enumerate each level by letting

$$(5.1) \quad T_\alpha = \{\tau(\alpha, \xi) : \xi < \omega_1\} \text{ for all } \alpha < \omega_2.$$

In this situation we identify branches with functions from  $\omega_2$  to  $\omega_1$  that are induced by the enumerations of the levels. If  $A \subseteq \omega_2$  is unbounded and  $b : \omega_2 \rightarrow \omega_1$  is an  $\omega_2$ -branch through  $T$  then we say that  $b$  *oscillates on*  $A$  if for all  $\alpha < \omega_2$  and all  $\zeta < \omega_1$  there is  $\beta > \alpha$  in  $A$  and  $\xi > \zeta$  such that  $b(\beta) = \xi$ .

**9 Lemma.** *Assume that  $T$  is an  $\omega_2$ -Kurepa-tree with an enumeration  $\tau(\alpha, \xi)$  ( $\alpha < \omega_2$ ,  $\xi < \omega_1$ ) as in (5.1) and  $A_\iota$  ( $\iota < \omega_2$ ) are  $\aleph_2$ -many unbounded subsets of  $\omega_2$ . Then there is an  $\omega_2$ -branch  $b$  through  $T$  that oscillates on every  $A_\iota$  ( $\iota < \omega_2$ ).*

*Proof.* Assume not, then for every  $\omega_2$ -branch  $b$  there is  $\iota_b < \omega_2$  and there are  $\alpha_b < \omega_2$ ,  $\zeta_b < \omega_1$  such that

$$b \upharpoonright (A_{\iota_b} \setminus \alpha_b) \subseteq \{\tau(\alpha, \xi) : \alpha \in A_{\iota_b} \setminus \alpha_b, \xi < \zeta_b\}.$$

By a cardinality argument we can find  $\aleph_3$ -many branches  $b$  such that  $\iota_0 = \iota_b$ ,  $\alpha_0 = \alpha_b$  and  $\zeta_0 = \zeta_b$ . But then each of these branches is a different branch through the tree

$$T_0 = \{\tau(\alpha, \xi) : \alpha \in A_{\iota_0} \setminus \alpha_0, \xi < \zeta_0\}.$$

$T_0$  has countable levels but  $\aleph_3$ -many branches, a contradiction.  $\square$

## 6. DESTROYING REGRESSIVE KUREPA-TREES WITH MM

We introduce a simplified notation for the following arguments: if  $f : t \rightarrow \omega_1$  for some  $t \in \text{Nm}$  and  $\pi \in [t]$  then we let

$$\sup^{(f)}(\pi) = \sup_{n < \omega} f(\pi \upharpoonright n).$$

If  $b : \omega_2 \rightarrow \omega_1$  is an  $\omega_2$ -branch and  $x \in \omega_2^{<\omega}$  then  $b(x)$  really denotes the countable ordinal  $b(\text{tag}(x))$ .

---

<sup>2</sup>A  $\kappa$ -Kurepa-tree  $T$  is called *slim* if  $|T_\alpha| \leq |\alpha|$  for all  $\alpha < \kappa$ .



**10 Lemma.** *Assume that  $T$  is an  $\omega_2$ -Kurepa-tree and  $\mathcal{B}$  is the set of branches. Let  $\tau(\alpha, \xi)$  ( $\alpha < \omega_2$ ,  $\xi < \omega_1$ ) be an enumeration as in (5.1). Then in the Namba extension  $V^{\text{Nm}}$  there is a sequence*

$$\Delta_G = \langle \delta_n^G : n < \omega \rangle$$

*cofinal in  $\omega_2^V$  such that*

$$\dot{E}_{\mathcal{B}} = \{\sup^{(b)}(\Delta_G) : b \in \mathcal{B}\}$$

*is stationary relative to every stationary  $S \subseteq \omega_1$  in  $V$ , i.e.  $\dot{E}_{\mathcal{B}} \cap S$  is stationary for all stationary  $S \subseteq \omega_1$  in the ground model.*

*Proof.* Assume that  $\dot{C}$  is an Nm-name for a club in  $\omega_1$ ,  $S \subseteq \omega_1$  is a stationary set in  $V$  and  $t_0$  a condition in Nm. Our goal is to find a condition  $t_3 \leq t_0$  and an ordinal  $\xi_0 \in S$  such that  $t_3 \Vdash \xi_0 \in \dot{C} \cap \dot{E}_{\mathcal{B}}$ . By a fusion argument similar to the ones in [11, p.188], we construct a condition  $t_1 \leq t_0$  and a coloring  $f : t_1 \rightarrow \omega_1$  such that

- (1)  $f$  increases on chains, i.e. if  $v \subsetneq x$  are elements of  $t_1$  then  $f(v) < f(x)$ .
- (2) if the height of  $x$  in  $t_1$  is odd and  $x$  is above the trunk then there is  $\zeta < \omega_1$  such that

$$|\{x \frown \beta \in t_1 \mid f(x \frown \beta) = \xi\}| = \aleph_2 \text{ for all } \xi > \zeta,$$

i.e. each ordinal in a final segment of  $\omega_1$  has  $\aleph_2$ -many preimages in the set  $\text{Suc}_{t_1}(x)$ .

- (3) if  $G \subseteq \text{Nm}$  is generic with  $t_1 \in G$  and  $\pi : \omega \rightarrow \omega_2^V$  is the corresponding Namba-sequence then  $\sup^{(f)}(\pi) \in \dot{C}[G]$ .

Given the condition  $t_1$ , we apply Lemma 9 to find a branch  $b$  that oscillates on all sets  $\text{Suc}_{t_1}(x)$  ( $x \in t_1$ ). Using (1) and (2), we thin out again to get a condition  $t_2 \leq t_1$  with the following property:

- (4) if  $v \subsetneq x \subsetneq y$  is a chain in  $t_2$  above the trunk and the height of  $x$  is odd then  $f(v) < b(x) < f(y)$ .

Note that in particular (2) can be preserved by passing to the condition  $t_2$ , so we may assume that  $t_2$  has properties (1)-(4). Let us also assume for notational simplicity that the height of  $\text{tr}(t_2)$  is even. The next step is to find  $t_3 \leq t_2$  and  $\xi_0 \in S$  such that

- (5)  $\sup^{(f)}(\pi) = \xi_0$  for all branches  $\pi$  in  $[t_3]$ .

To find  $t_3$  and  $\xi_0$ , we define a game  $\mathbb{G}(\gamma)$  for every limit  $\gamma < \omega_1$ . Fix a ladder sequence  $l(\gamma) = (\gamma_n : n < \omega)$  for each such  $\gamma$ . The game  $\mathbb{G}(\gamma)$  is played as follows:

I	$\alpha_0$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\dots$
II	$\beta_0$	$\beta_1$	$\beta_2$	$\beta_3$	$\dots$

where for all  $n < \omega$

- $\alpha_n < \beta_n < \omega_2$ ,
- $s_n = \text{tr}(t_2)^\frown(\beta_i : i \leq n) \in t_2$  and
- $f(s_n) \in (\gamma_n, \gamma)$  whenever  $n$  is even.

II wins if he can make legal moves at each step, so the game is determined.

**10.1 Claim.** *II wins  $\mathbb{G}(\gamma)$  for club many  $\gamma$ 's.*

*Proof.* Assume not, then there is a stationary  $U \subseteq \omega_1$  such that player I wins  $\mathbb{G}(\gamma)$  for each  $\gamma \in U$  via the strategy  $\sigma_\gamma$ . Now pick a countable elementary  $N$  such that  $\xi = N \cap \omega_1 \in U$  and  $t_2, f, l, U \in N$ .

A ladder sequence  $l(\xi) = (\xi_n : n < \omega)$  converging to  $\xi$  is given and we define a sequence  $(\beta_n : n < \omega)$  inductively as follows: let  $\beta_n$  be the least

$$\beta > \sup_{\gamma \in U} \sigma_\gamma(\beta_i : i < n)$$

such that

- $s = \text{tr}(t_2)^\frown(\beta_i : i < n)^\frown \beta \in t_2$  and
- $f(s) \in (\xi_n, \xi)$  whenever  $n$  is even.

Such a  $\beta$  exists in  $N$  by (2) and elementarity. Note that  $(\beta_n : n < \omega)$  is a possible record of moves for player II if player I goes along with the strategy  $\sigma_\xi$ . But II obviously wins the game  $\mathbb{G}(\xi)$  if the sequence  $(\beta_n : n < \omega)$  is played, a contradiction. This proves the claim.  $\square$

Given the claim, pick  $\xi_0 \in S$  above all  $b(x)$  ( $x \subseteq \text{tr}(t_2)$ ) such that II wins the game  $\mathbb{G}(\xi_0)$ . Now we can easily find a condition  $t_3 \leq t_2$  with property (5).

If we fix a generic  $G \subseteq \text{Nm}$  with  $t_3 \in G$  and let  $\pi_G : \omega \longrightarrow \omega_2^V$  be the corresponding Namba-sequence, we can define  $\delta_n^G = \pi_G(2n+1)$  and  $\Delta_G = \langle \delta_n^G : n < \omega \rangle$ . Then we have

- (6)  $\sup^{(f)}(\Delta_G) \in \dot{C}[G]$  by (3),
- (7)  $\sup^{(f)}(\Delta_G) = \xi_0$  by (5) and
- (8)  $\sup^{(f)}(\Delta_G) = \sup_{n < \omega} b(\delta_n^G) = \sup^{(b)}(\Delta_G)$  by (4).

But this finishes the proof since

$$\xi_0 \in \dot{C}[G] \cap \dot{E}_{\mathcal{B}}[G] \cap S.$$

$\square$

**11 Corollary.** *Assume that  $T$  is an  $\omega_2$ -Kurepa-tree and  $\mathcal{B}$  the set of ground model branches through  $T$ . Then  $\mathcal{B}$  is stationary over  $T$  in the Namba extension.*

Finally we get the main result for  $\omega_2$ . We will prove a more general version of this in Theorem 13.

**12 Theorem.** *There are no  $\omega_1$ -regressive  $\omega_2$ -Kurepa-trees under MM.*

*Proof.* Assume that  $T$  is an  $\omega_1$ -regressive  $\omega_2$ -Kurepa-tree and that

$$\tau(\alpha, \xi) \ (\alpha < \omega_2, \xi < \omega_1)$$

is an enumeration as in (5.1). Look at the iteration  $\mathbb{P} = \text{Nm} * \text{CS}(\dot{E}_{\mathcal{B}})$ , where  $\text{CS}(\dot{E}_{\mathcal{B}})$  shoots a club through the set  $\dot{E}_{\mathcal{B}}$  from the statement of Lemma 10. The poset  $\mathbb{P}$  preserves stationary subsets of  $\omega_1$  by the fact that  $\dot{E}_{\mathcal{B}}$  is stationary relative to every stationary set in  $V$ . But we have that  $\dot{E}_{\mathcal{B}}$  is club in  $V^{\mathbb{P}}$ , so we can use MM to get a sequence  $\Delta = \langle \delta_n : n < \omega \rangle$  converging to  $\delta < \omega_2$  such that

$$\{\sup^{(b)}(\Delta) : b \text{ is a } \delta\text{-sequence in } T_\delta\}$$

is club in  $\omega_1$ . Using Lemma 4, we see that  $T_\delta$  is definitely stationary over  $T \restriction \Delta$ . So  $T_\delta$  is stationary over  $T_{<\delta}$  by Remark 3. Since  $\text{cf}(\delta) = \omega$ , this contradicts the fact that  $T$  is  $\omega_1$ -regressive.  $\square$

## 7. LARGER HEIGHTS

Starting from Theorem 12, we generalize the result to weak Kurepa-trees in all uncountable regular heights.

**13 Theorem.** *Under MM, there are no  $\omega_1$ -regressive weak  $\lambda$ -Kurepa-trees for any uncountable regular  $\lambda$ .*

*Proof.* Since PFA destroys weak  $\omega_1$ -Kurepa-trees (see [1]), we may assume that  $\lambda$  is at least  $\omega_2$ . Now assume that  $T$  is an  $\omega_1$ -regressive weak  $\lambda$ -Kurepa-tree and let  $\mathcal{P} = \text{Col}(\omega_2, \lambda)$  be the usual  $\omega_2$ -directed collapse. Note that  $\mathcal{P}$  has the  $\lambda^+$ -cc, because  $\lambda^{\omega_1} = \lambda$  holds under MM (see [2]). So the tree  $T$  has a cofinal subtree  $T^*$  in  $V^{\mathcal{P}}$  that is an  $\omega_1$ -regressive weak  $\omega_2$ -Kurepa-tree. By throwing away some nodes if necessary, we may assume that  $T^*$  has the property that

$$(7.1) \quad T_x^* = \{y \in T^* : x \leq_T y\} \text{ has } \aleph_3\text{-many branches for all } x \in T^*.$$

Now we define an  $\omega_2$ -directed forcing  $\mathcal{Q}$  in  $V^{\mathcal{P}}$  that shoots an actual  $\omega_2$ -Kurepa-subtree through the tree  $T^*$ : conditions of  $\mathcal{Q}$  are pairs of the form  $(S, B)$ , where

- (1)  $S$  is a downward-closed subtree of  $T^*$  of height  $\alpha + 1$  for some ordinal  $\alpha < \omega_2$ ,
- (2)  $|S| \leq \omega_1$ ,
- (3)  $B$  is a nonempty set of branches cofinal in  $T^*$  and  $|B| \leq \aleph_1$ ,
- (4)  $b \restriction (\alpha + 1) \subseteq S$  for all  $b \in B$ .

We let  $(S_0, B_0) \geq_Q (S_1, B_1)$  if  $S_0 = S_1 \restriction \text{ht}(S_0)$  and  $B_0 \subseteq B_1$ .

If  $X \subseteq Q$  is a set of mutually compatible conditions of size  $\leq \aleph_1$  then we let  $S_X$  and  $B_X$  be the unions over the first respectively second coordinates of  $X$ . Now  $S_X$  can be end-extended to a tree  $\bar{S}_X$  of successor height by extending at least the cofinal branches in the non-empty set  $B_X$ . But then  $(\bar{S}_X, B_X)$  is a condition stronger than every condition in  $X$ , hence  $Q$  is  $\omega_2$ -directed-closed. An easy cardinality argument shows that  $Q$  has the  $\aleph_3$ -cc because  $2^{\aleph_1} = \aleph_2$  holds in  $V^P$ . It is now straightforward that a generic filter  $H \subseteq Q$  will produce an  $\omega_2$ -tree in the first coordinate which is  $\omega_1$ -regressive since it is a subtree of the original tree  $T$  and notice that  $P$  and  $Q$  both preserve uncountable cofinalities. On the other hand, a density argument using (7.1) shows that the set

$$\mathfrak{B} = \bigcup \{B : \text{there is } S \text{ such that } (S, B) \in H\}$$

has cardinality  $\aleph_3$ , so  $H$  induces an  $\omega_1$ -regressive  $\omega_2$ -Kurepa-tree. The composition of two  $\omega_2$ -directed-closed forcings is again  $\omega_2$ -directed-closed and it was mentioned in the introduction that  $\omega_2$ -directed-closed forcings preserve MM, so we have the situation:

- $V^{P*Q} \models \text{MM}$
- $V^{P*Q} \models \text{“there is an } \omega_1\text{-regressive } \omega_2\text{-Kurepa-tree.”}$

But this contradicts Theorem 12. □

Using Theorem 5, we have

**14 Corollary.** *MM is sensitive to  $\lambda$ -closed forcing algebras for arbitrarily large  $\lambda$ .*

## REFERENCES

- [1] James Baumgartner. Applications of the Proper Forcing Axiom. In K. Kunen and J.E. Vaughan, editors, *Handbook of set-theoretic topology*, pages 913–959. North-Holland, 1984.
- [2] M. Foreman, M. Magidor, and S. Shelah. Martin’s maximum, saturated ideals, and nonregular ultrafilters I. *Annals of Mathematics*, 127:1–47, 1988.
- [3] Moti Gitik and Saharon Shelah. On certain indestructibility of strong cardinals and a question of Hajnal. *Archive for Mathematical Logic*, 28:35–42, 1989.
- [4] Thomas Jech. *Set Theory*. Perspectives in Mathematical Logic. Springer-Verlag, 1997.
- [5] Akihiro Kanamori. *The Higher Infinite*. Perspectives in Mathematical Logic. Springer-Verlag, 1997.
- [6] Bernhard König. Local Coherence. *Annals of Pure and Applied Logic*, 124:107–139, 2003.
- [7] Bernhard König and Yasuo Yoshinobu. Fragments of Martin’s Maximum in generic extensions. *Mathematical Logic Quarterly*, 50:297–302, 2004.

- [8] Paul Larson. Separating stationary reflection principles. *Journal of Symbolic Logic*, 65:247–258, 2000.
- [9] Paul Larson. The size of  $\tilde{T}$ . *Archive for Mathematical Logic*, 39:541–568, 2000.
- [10] Richard Laver. Making the supercompactness of  $\kappa$  indestructible under  $\kappa$ -directed closed forcing. *Israel Journal of Mathematics*, 29:385–388, 1978.
- [11] W. Hugh Woodin. *The Axiom of Determinacy, Forcing Axioms, and the Non-stationary Ideal*. Walter de Gruyter & Co., Berlin, 1999.

BOISE STATE UNIVERSITY  
DEPARTMENT OF MATHEMATICS  
BOISE ID 83725-1555  
USA

*E-mail address:* bkoenig@diamond.boisestate.edu

GRADUATE SCHOOL OF INFORMATION SCIENCE  
NAGOYA UNIVERSITY  
FUROCHO, CHIKUSA-KU, NAGOYA 464-8601  
JAPAN

*E-mail address:* yosinobu@math.nagoya-u.ac.jp